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# A STUDY IN AUCTION THEORY AND A SCENARIO IN CHEATING IN ENGLISH AUCTIONS

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# Abstract

Auction theory is a part of game theory that studies human behavior in auction markets and resulting market outcomes. Research in this area, which is admittedly lacking, suggests that cheating is a risk that players sometimes find profitable to take. Within this body of literature, researchers are inevitably beginning to investigate cheating scenarios. In order to model real-life problems using the tools that these theories give us, a model of a cheating scenario in an English auction is proposed in this thesis. We find Nash equilibrium for this model by modifying the settings of the scenario. These Nash equilibria "make sense" in terms of reacting in the right way to changes in model parameters. As we generalize the situation, we find that the results follow well-known results in auction theory; therefore, we can verify that our proposed model works for this scenario.

# Dedication

To the innocent and precious lives taken away in the downing of the Ukrainian plane in January 2020.

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# Chapter 1

# Introduction and Background

## 1.1 Introduction

A considerable amount of economic transactions are conducted through auctions. Auctions have been used for the sale of a variety of objects. Nowadays, numerous commodities, from famous people's personal items to art supplies, are sold via auctions. Moreover, auctions provide a well-defined transaction environment for testing economic and mathematical theories, such as a game theory with incomplete information.

This thesis aims to model scenarios in auctions because such is reflective of reality. Thus, this thesis converts real-life problems into mathematical settings and then examines them via a game theory perspective and offers solutions. We will begin by discussing auction theory, describe what an auction is, introduce the types of auctions that are mainly used, and discuss how their processes work. However, beforehand, we discuss game theory, which has provided us with the tools to study and analyze auctions and our scenarios.

## 1.2 Game Theory

Game theory is the science of studying strategic decision-making in competitive social situations. Games in this setting usually refer to events whose results depend on the actions of decision-makers; these decision-makers are also called players in the game. Research in game theory helps us model real-world scenarios onto games and use the theoretical framework provided by game theory to study and find optimal strategies for the players.

A strategy is a complete plan of actions that a player will follow, having information about possible situations in the game. The goal of decision-makers in the game is to make their outcome better, i.e., by choosing a strategy that will maximize their payoff or minimize their loss.

In a strategic form game where players are interacting, each player has a set of possible strategies. Here, players are affected not only by their own decisions but also by the actions of other players in the game[7].

**Definition 1.1:** A strategic form game includes:

1) A set of players, $N = \{1, 2, \ldots, n\}$

2) A set of strategies for each player

3) A payoff function that assigns a payoff to each set of strategies representing players' evaluations over the set of possible joint actions.

Two games can be regarded as "strategically equivalent" if, for every player, if strategic variables, such as best responses of players, are the same for each player.

### 1.2.1 Non-cooperative Game Theory

There are several classifications of games in game theory, such as symmetric/asymmetric, sequential/simultaneous, cooperative/non-cooperative [6], etc. In cooperative game theory, we assume players may make binding contracts. Hence, the emphasis is on how the grand coalition of all players should divide up that which they can collectively gain. In non-cooperative game theory, on the other hand, the players cannot or will not make binding contracts. Hence, the emphasis is on how rational players interact for their own benefit.

John Nash, known for his Nobel prize for economics and his work in game theory, studied non-cooperative games in 1951 in Annals of Mathematics. His solution concept for such games (Nash equilibrium or non-cooperative equilibrium) tries to predict players' strategies and payoffs in such games.

### 1.2.2 Nash Equilibrium

As we analyze different scenarios in auctions, we are interested in modeling them to determine the optimal strategies for our players, which gives the player the best outcome, given the player's input and the game settings. Sometimes, when there are multiple players in the game in the same situation, we are also interested in finding the strategy that works best for all those players simultaneously, if there is any.

That is why we want to introduce a concept from game theory:

A strategic equilibrium (or Nash equilibrium) for an $n$-person strategic form game is a set of strategies, one for each player, such that no player can improve his expected payoff just by switching only his own strategy. More formally, we can have the following definition:

**Definition 1.2:** Suppose $\mathcal{S}_i$ is the set of strategies for player $i$. Suppose $h^i(S_1, \ldots, S_n)$ is equal to payoff to player $i$ if player one plays $S_1$, player two plays $S_2$,..., and player $n$ plays $S_n$. Then, $(S_1^*, ..., S_n^*)$ is a Nash equilibrium if:

$$h^i(S_1^*, \ldots, S_n^*) \geq h^i(S_1^*, \ldots, S_{i-1}^*, S_i, S_{i+1}^*, ..., S_n^*) \quad \text{for all} \quad S_i \in \mathcal{S}_i, i = 1, ..., n \tag{1.1}$$

In short, a Nash equilibrium is a stability concept, which is often used to predict the outcome of non-cooperative games. In fact, it is by far the most commonly used solution concept in game theory.

## 1.3 Auction Theory

The kind of auction we are working on is a one-to-many transaction, where we have one seller selling a good to a few buyers. The seller has an estimated value of the good he wishes to sell and wants to sell it at a price that will profit him the most. He can set specific regulations for the auction, such as setting an entry fee. Each buyer who wants to attend the auction must pay this fee in advance, regardless if he wins or loses [1]. The buyers, on the other hand, each have their own private valuation of the item. Hence, buyer $i$ will submit a bid, $b_i$, based on his private valuation $\theta_i$, with the goal of buying the item for a price that is less than or equal to his valuation so that he will make some positive profit.

### 1.3.1 Types of Auction

Auctions can be categorized into "open" auctions and "sealed-bid" auctions [3]. In "open" auctions, each player can see what the other bidders are submitting, and they can adjust their bids accordingly. The bidder may start with an initial bid and change it throughout the entire process until changing his last bid is no longer profitable. Auctions can be "ascending" or "descending." In the former, the seller or auctioneer starts with a low price and raises it until only one bidder remains; in a descending auction, the seller starts with a very high price and lowers it. Once the bidder declares that he will pay the price, he wins the object.

On the other hand, a "sealed-bid" auction allows the players to submit only one bid at the beginning of the auction process without having any information about other players' submissions. They will wait for the process to finish, after which the auctioneer will tell them who placed the highest bid and how much they have to pay.

A few of the most well-known and commonly used types of auctions include the following [4, 5]:

#### 1.3.1.1 English Auction

The English auction is the most well-known and perhaps oldest auction in use. The auctioneer, who can be the seller or not, will begin setting the price at a low amount and gradually increase it by small increments. Each bidder, who finds the price still profitable, will give the auctioneer a sign, primarily by nodding or raising a hand. The auctioneer will keep raising the price until no one else wants the good at that price. He will end the auction at this point and give the goods to the winning bidder after he checks that no one is willing to raise the price. There might be some regulations the seller or auctioneer set up for the auction that may dictate the amount the winner is required to pay, but most often, that amount is the same as what he bid.

#### 1.3.1.2 Dutch Auction

Similar to an English auction, the Dutch auction is also a type of open auction. However, instead of increasing the price gradually, the auctioneer starts with a relatively high amount and keeps lowering it. Initially, the auctioneer would begin at a price so high that no buyer would find it profitable to buy the item at that price. Then he starts decreasing the bid until one bidder shows interest in purchasing the good. Because of this, the Dutch auction is referred to as an open descending price auction.

#### 1.3.1.3 First-Price Sealed-bid Auction

In this type of auction, bidders submit their bids, calculated based on their initial valuations of the good, in a sealed envelope. When all bids have been submitted, the auctioneer will open the bids and call the buyer with the highest bid the winner. By paying the amount bid, the winner will own the item.

#### 1.3.1.4 Second-Price Sealed-bid Auction

Second-price sealed-bid auctions, also known as "Vickrey auctions" after William Vickrey, are conducted just as in first-price auctions, where bidders submit sealed envelopes for each bid at the beginning of the auction. The only difference in sealed-bid auctions is that in the second-price auction, the winning bidder, who has submitted the highest bid, will pay the amount equal to the second-highest bid.

In comparison to open auctions, sealed-bid auctions are easier to analyze since they require little communication. At each auction, sellers or auctioneers may set some rules that are followed during the auction. For instance, there can be a "reservation price" for the sale. By setting such a variable, each player's bid has to be at least equal to the reservation price if they want to participate in the sale.

Besides these common forms of auctions, there are a wide variety of auctions taking place to buy and sell goods around the world. Each kind of auction can be used to sell and buy any type of good. The following are other forms of auctions: Reserve Auction, Minimum Bid Auction, The Yankee Auction, Multi-

Parcel Auction, etc.

## 1.4 Evaluations and Bidding Strategies

In the basic auction model, each bidder has a private valuation of the good, known only to themselves and not to the other bidders. Bidders try to submit bids according to their private values in a way that they can maximize their profit from buying that item. On the other hand, sellers try to sell the item to a bidder whose bid can maximize their own profit.

Let us consider an auction with one seller and a set $N$ of buyers in the game, and there is one object for sale. Bidder $i, i \in N$, has a private value $\theta_i$ for the object, which is the highest price he is willing to pay for the item. Each private value is distributed on an interval, $[0, \infty)$, independently by an increasing distribution function $F$. We assume $F$ and $N$ are known to all players.

In general, we believe that each player is risk-neutral, meaning they just want to maximize their expected profit. In each auction, each bidder realizes that the more he bids, the higher his chances of winning, and at the same time, the lower his profit. It is also apparent that it is not profitable for players to bid higher than their valuations. As a result, we can assume that a player with a private valuation of zero will not submit a positive bid.

Let us consider that the game is symmetric, meaning all the players have the same distribution function, $F$, but their private values are still independent and known only to themselves. A strategy for this game is a function that takes the player's private value, $\theta_i$, and gives a value, $b_i(\theta_i)$, to submit a bid at that price. From the above, we conclude $b_i(\theta_i) \leq \theta_i$.

### 1.4.1 First Price Sealed-bid Auction

In a first-price sealed-bid auction, each bidder submits a bid and pays the amount equal to his bid if he wins. The payoff for player $i$ after submitting a bid $b_i(\theta_i)$ will be equal to:

$$\begin{cases} \theta_i - b_i(\theta_i) & \text{if b}_i \text{ is the highest bid} \\ 0 & \text{Otherwise} \end{cases} \tag{1.2}$$

A player's profit is equal to the amount his valuation is greater than the price he has paid. Under these rules, it is clear that it is not profitable for players to bid their true values since it would result in zero profit. By bidding slightly below their values, they might be able to make a positive profit if they win the bidding. Player $i$ will win the auction if his bid is higher than that of other $n-1$ players. Therefore, the optimal strategy for player $i$ is [4]:

$$b_i(\theta_i) = \theta_i - \int_0^{\theta_i} \frac{F(y)^{(n-1)}}{F(\theta_i)^{(n-1)}} dy \tag{1.3}$$

The following proposition verifies that it is correct:

**Proposition 1.4.1** In an $n$-player first-price auction, symmetric equilibrium strategies are given by

$$b(\theta) = E[Y^*|Y^* < \theta_i] = \theta_i - \int_0^{\theta_i} \frac{F(y)^{(n-1)}}{F(\theta_i)^{(n-1)}} dy \tag{1.4}$$

where $Y^*$ is the highest of $n-1$ independently drawn bids.

**Proof:**

We want to show that in a symmetric game, meaning all the players are following the same strategy, the best strategy is $b(\theta) = E[Y^*|Y^* < \theta_i]$. To prove this, let us suppose all players except one, player $i$, follow this strategy. We will show it is also the optimal strategy for player $i$ to follow it.

Note that this function is a continuous and increasing function, therefore, a player with the highest private value will submit the highest bid. Suppose player $i$'s private value is $\theta$ and instead of $b(\theta)$ he decides to submit $b_1$. We set $z = b^{-1}(b_1)$, so we can assume player $i$ is bidding $b(z) = b_1$.

We want to show it is not optimal for player $i$ a bid $b_1$, no matter $b_1 \geq b(\theta)$ or $b_1 < b(\theta)$. Therefore, we should show whether $z \geq \theta$ or $z < \theta$, $E(b(\theta), \theta) \geq E(b(z), \theta)$, where $E(x, y)$ is the expected payoff of the player having bid $x$ and private value $y$.

Thus his expected payoff by bidding according to $b(z)$ would be:

$$\begin{aligned} E(b_1, \theta) &= F^{n-1}(z)[\theta - b(z)] \\ &= F^{n-1}(z)[\theta - E[Y^*|Y^* < \theta]] \\ &= F^{n-1}(z)\theta - \int_0^z y(n-1)F^{n-2}(y)f(y)dy \\ &= F^{n-1}(z)\theta - F^{n-1}(z)z + \int_0^z F^{n-1}(y)dy \\ &= F^{n-1}(z)(\theta - z) + \int_0^z F^{n-1}(y)dy \end{aligned} \tag{1.5}$$

On the other hand, his expected payoff by bidding according to (1.4) would be:

$$\begin{aligned}
E(b(\theta),\theta) &= F^{n-1}(\theta)[\theta - b(\theta)] \\
&= F^{n-1}(\theta)[\theta - E[Y^*|Y^* < \theta]] \\
&= F^{n-1}(\theta)(\theta - [\theta - \frac{\int_0^\theta F^{n-1}(y)dy}{F^{n-1}(\theta)}]) \\
&= \int_0^\theta F^{n-1}(y)dy
\end{aligned} \tag{1.6}$$

Therefore,

$$\begin{aligned}
E(b(\theta),\theta) - E(b(z),\theta) &= \int_0^\theta F^{n-1}(y)dy - [F^{n-1}(z)(\theta - z) + \int_0^z F^{n-1}(y)dy] \\
&= F^{n-1}(z)(z-\theta) - \int_\theta^z F^{n-1}(y)dy \geq 0
\end{aligned} \tag{1.7}$$

Thus, $b(\theta)$ is also the optimal strategy for player $i$, and we can rewrite it as:

$$\begin{aligned}
b(\theta) &= \frac{1}{F^{n-1}(\theta)} \int_0^\theta yf(y)dy \\
&= \theta - \frac{\int_0^\theta F^{n-1}(\theta)dy)}{F^{n-1}(\theta)} \\
&= \theta - \int_0^\theta \frac{F^{n-1}(\theta)dy}{F^{n-1}(\theta)}
\end{aligned} \tag{1.8}$$

**Proposition 1.4.2** In a first-price auction with $n$ players whose private values are uniformly distributed on $[0,1]$, an optimal strategy is for players to bid according to:

$$b(\theta) = \frac{n-1}{n}\theta \tag{1.9}$$

**Proof:**

Private values are uniformly distributed on $[0, 1]$. We know the distribution function of this uniform distribution is $F(x) = x$; Therefore, by substituting $F(x)$ in equation (1.6) we will get:

$$
\begin{aligned}
b(\theta) &= \theta - \int_0^\theta \frac{y^{(n-1)}}{\theta^{(n-1)}} dy \\
&= \theta - \frac{1}{\theta^{(n-1)}} [\frac{y^n}{n}]_0^\theta \\
&= \theta - \frac{\theta}{n} \\
&= \frac{n-1}{n}\theta
\end{aligned} \tag{1.10}
$$

### 1.4.2 Second Price Sealed-bid Auction

In the second-price sealed-bid auction, the bidder with the highest bid will win the game, but he would pay the amount equal to the second-highest bid. Contrary to a first-price auction, it is a dominant strategy for players to bid their true valuation.

The general strategy here is much simpler than the first-price auction. The profit function for player $i$ will be:

$$
\begin{cases}
\theta_i - (\text{the second-highest bid}) & \text{if } b_i \text{ is the highest bid} \\
0 & \text{Otherwise}
\end{cases} \tag{1.11}
$$

**Proposition 1.4.3** In a second-price auction, it is a weakly dominant strategy to bid according to

$$b(\theta) = \theta$$

**Proof:**

Suppose we have $n$ players in a second-price auction. The private value of player $i, i \in 1, 2, \ldots, n$ is $\theta_i$, and he bids according to $b_i(\theta_i)$. Let us define $b^*$ as the highest bid out of $b_j(\theta_j)$ for $j = 1, 2, ..., n, j \neq i$.

The associated bid for player $i$ can be either greater than, equal to, or less than his private value. First, let us suppose that $b_i(\theta_i) > \theta_i$, therefore, there will be 3 possible cases for player $i$'s bid:

1. $b^* > b_i(\theta_i) > \theta_i$

   In this case, as the associated bid is less than the highest bid, player $i$ will not win. So the expected payoff will be zero. Setting $b_i(\theta_i) = \theta_i$ would also pay off zero, so nothing is lost by bidding $\theta_i$ instead of $b_i(\theta_i)$.

2. $b^* < \theta_i < b_i(\theta_i)$

   If both private value and its associated bid are higher than $b^*$, player $i$ will win the bidding. However, the expected payoff, $\theta_i - b_i(\theta_i)$, will be negative since $b_i(\theta_i) > \theta_i$, therefore, he would win and gain a nonnegative payoff by bidding according to $b_i(\theta_i) = \theta_i$. Another way to put this is that, the second highest bid will be $b^*$, whether player $i$ bids $\theta_i$ or $b_i(\theta_i)$. Thus, again nothing is lost by bidding $\theta_i$ instead of $b_i(\theta_i)$.

3. $b_i(\theta_i) > b^* > \theta_i$

   Here, player $i$ will win the bidding, but the expected payoff will be $\theta_i - b_i(\theta_i)$ which will be negative. Therefore, although bidding equal to his valuation will not make him win the auction, the expected payoff of zero is better than a negative payoff.

A similar argument shows that player $i$ will do better by bidding $\theta_i$ rather than $b_i(\theta_i)$ when $b_i(\theta_i) < \theta_i$.

Hence, it is a dominant strategy for players in a second-price auction to bid their own private value. **QED**

### 1.4.3 English Auction

In the open ascending auction, or English auction, each player should decide at what price they want to announce their bids. This price should be conditioned on no other players having called this yet and bringing a positive profit for the buyer, which means his private value minus the amount he bid should be positive. Ultimately, the bidder with the highest bid will win the item at the price he has called out and pay that amount.

The game is equivalent to second-price sealed-bid auctions when values are private; however, this equivalence is weak since they are not strategically equivalent [4]. The English auction offers information about when other bidders drop out, and by observing this, a player may be able to infer something about his private valuation of the item. However, if the values are private, this information is not helpful. Therefore, in an English auction, it is not optimal to stay in after the price has exceeded the value (since it will lead to a loss), nor is it optimal to drop out before the price reaches the value (since it will result in losing potential gains). Likewise, in a second-price auction, it is best to bid the private value. So, having private values, the optimal strategy in both auctions is to bid up to or stay until the private value is reached.

### 1.4.4 Dutch Auction

In the open descending auction, on the other hand, it is the best strategy for a bidder to stay in the game until the auctioneer reaches the price at which buying the item is profitable. The seller will continue decreasing the price from a high point until a player calls it. This player had the highest valuation of the object since it was profitable for him to buy it at that price, while at the same time, others could not afford to buy it and were waiting for a lower price.

In a strategic sense, the Dutch (open descending) auction is equivalent to the first-price sealed-bid auction. Strategic equivalence means that for every strategy in one game, a player has a strategy in the other game, resulting in the same outcomes. In a first-price sealed-bid auction, the bidder's strategy is a function of his private valuation without knowing anything else about other players' valuations and bids. Although the Dutch auction differs from a sealed bid auction in that it takes place in the open, it provides no useful information to bidders. The only available information is that some bidders have agreed to purchase at the current price, but that ends the auction. Therefore, a player has an equivalent strategy in the Dutch auction and a first-price auction, and the player with the highest bid will win the item and be obligated to pay the amount equal to that bid.

# Chapter 2

# Cheating in Auctions

## 2.1 Introduction

As auctions become increasingly popular, more cheating occurs, and cheating can occur in any auction type. Furthermore, any player (bidder, seller, or auctioneer) who wishes to increase his expected profit may cheat. A bidder who is a cheater tries to change the game to increase the probability of winning and/or winning more by bidding less.

Most of the time, cheating in online auctions is easier than in the auctions where players are present. This is why auction frauds constitute a large part of Internet fraud [2]. Nevertheless, cheating in auctions is a fraud that often leaves few or no tracks after the event has occurred.

In an auction, the seller tries to sell the item at a price that will maximize his expected revenue. On the other hand, each buyer tries to buy the item at a price low enough to maximize his net payoff. If any of them decides to cheat, they may use illegal or unethical means to achieve these goals.

## 2.2 Types of Cheating

Both bidders and sellers can cheat in an auction. Cheating in this environment consists of Shill Bidding, Bid Shading, Multiple Bidding, False Bidding, and Rings [2, 8]. In addition, several types of cheating can occur simultaneously in a single auction. For instance, the seller can submit a false bid, and at the same time, some of the bidders can form a Ring.

A bid shading occurs when a corrupt bidder checks other players' bids before submitting his own, using unfair ways and adjusting his bid accordingly to pay less.

Multiple bidding is when a cheating player hires some fake bidders or plays with numerous fake identities in the game without the intention of buying the item in order to increase his chance of winning.

A subset of bidders can also form a ring in an auction and plan not to compete or raise the item's price.

One way that a seller or an auctioneer cheats in an auction is shill bidding. Shill bidding is when a cheating seller hires some agents to submit a fake bid to increase the item's price without the purpose of buying it. Another way for a seller to cheat is to submit a false bid. A false bid is a bid that the cheating seller submits in a second-price auction, just below the highest bid, to increase the amount that the winner would pay.

### 2.2.1 Sealed-bid Auctions

In sealed-bid auctions, since bidders submit their bids in sealed envelopes, most of the cheating occurs when one of the players attempts unfair ways to find out other players' submitted bids before the end of the game.

Therefore, in a first-price auction, where it is optimal for bidders to bid a portion of their valuation and pay the price bid, it would be profitable to know the other players' bids. By doing so, they could adjust their own bids based on others' bids so that they could win by paying the minimum amount needed and increase their profit.

From the seller's perspective, submitting a shill bid is not profitable. The seller will win if that bid is the highest, so he should cancel the game since it is not what he wanted. Otherwise, there will be no change in his expected payoff. Hence, the seller has no incentive to cheat in a first-price auction.

On the other hand, in second-price auctions, a bidder's dominant strategy is to bid his own value, so knowing others' bids will not help a cheating bidder. Therefore, a bidder has no incentive to cheat in a second-price auction. On the other hand, sellers can increase their profit if they can examine submitted bids before the auction ends and offer a fake bid just below the highest bid. As a result, the player with the highest bid will still win the item but will have to pay an amount almost equal to the amount he bid instead of the actual second-highest bid.

### 2.2.2 Open Auctions

The incentive an auctioneer or a seller has to cheat in a second-price auction is present in open auctions. The seller has an incentive to boost his revenue by raising the item's price. To do so, the seller will do shill bidding to increase the amount the winner should pay. However, cheating in an open auction may have some risks for the seller, which were not present in the second-price auction. In a second-price auction, cheating occurs after all players have made their bids, so they cannot change or withdraw their bids after the changes the seller has made, but, in an open auction, like in an English auction, the bidders may have the option to quit the game when they see the changes. If this happens, the fake bidder, whom the seller hired, will win the auction and must pay for the item, so the item remains unsold, which is a loss for the seller.

At the same time, open auctions also provide a profitable cheating opportunity for bidders. They can submit multiple bids on the same good, either by using fake identities or hiring fake bidders. Doing so would indirectly make other players quit the game. This happens because the fake bidders submit bids, usually higher than the cheating player's private value, to increase his winning chances. If these bids are high enough, other players will have no reason to stay in the game because increasing their bids will not bring positive profit for them after some point. Therefore, they will withdraw their bids and quit the game; after that, all the fake bidders hired by the cheating player will also withdraw and quit. Hence, the only player who remained in the game would be the cheater who could win the game with his lower bid, which will bring him a positive outcome.

Players can also cheat in an open auction by forming a ring, mostly in public auctions where players know other players' identities. They collude to keep the price of the item low by not bidding against each other. Members of a bidding ring profit by winning the item at a low price in the public auction and then re-auctioning it later in a private auction held by ring members and share the profit. The intention is to reduce competition by not competing against each other.

# Chapter 3

# Cheating in English Auctions and Model Results

## 3.1 The Model

Now that we have reviewed how players can conduct cheating in auctions, we model a scenario with $n$ players in an English auction. It is good to remember that in an English auction, players may understand or hear other players' bids during an auction, but they will not be able to adjust their own submitted bids based on others. The only action they might be able to take after submitting their bids, dependent on the auctioneer's decision at the beginning of the game, is to withdraw their bids [9].

We are specifically talking about cheating that occurs when players can hire fake bidders, and they will submit a bid that is higher than the person's actual bid and can be higher than everyone else's. If this bid is the highest submitted bid in the game, as we are in an English auction, other bidders (except for the player who hired the fake bidder) will quit the game, and the fake bidder would win. She will withdraw her bid afterward, so the next highest bid should be the winner-the cheating player. That player will win the game with positive profit while his actual bid was not the highest, so he could not automatically win the item without cheating.

The scenario that we are considering, however, is somewhat different. There is a probability $p^w$, known to all players, that a player can later retract his submitted bid. Therefore, by probability $(1 - p^w)$, the hired fake bidders might get caught and not be allowed to retract their bids, so they have to stay in the game and continue playing with their fake bids.

Let us define some terminology:

$\theta_i$: bidder $i$'s private value.

$b_i(\theta_i)$: bidder $i$'s actual bid based on $\theta_i$

$b_i^c(\theta_i)$: bidder $i$'s fake bid

$b_i^{c*}$: the value of $b_i^c(\theta_i)$ which maximizes the expected profit

$p^w$: the probability that players can withdraw their submitted bid

$a$: the reservation price set by the seller is the amount that seller values the item and is not willing to sell the object at a price lower than that

$F(x) = P(\theta_i \leq x)$: Distribution function for each player's private value

$F^*(x) = P({b^{c*}}_{-i} \leq x)$: Distribution function for the maximum of the other players' bids

Each (cheating) bidder can hire one fake bidder. Necessarily, the fake player's bid will be greater than her employer's actual bid. [If not, the submitted cheating bid would be of no use and will not change the game.] In this case, when the player $i$ decides to cheat and bring a fake bidder to the game, if his fake bidder wins the game, there would be two possibilities:

1) The first scenario is when the fake bidder can withdraw her fake bid, so the next highest remaining bidder, who is the cheating employer $i$, will win the game and pay his actual bid. As discussed before, for a rational player, his actual bid will be less than or equal to his valuation of the good, so his payoff will be a positive amount. The payoff function would be the probability of winning the game (his fake bid is greater than every other bid in the game) multiplied by the amount he would pay.

2) The second possibility is when the fake bidder cannot withdraw her bid after she committed cheating. Therefore, player $i$ has to pay the submitted fake bid. Thus, his payoff would be his private valuation of the good minus the fake bid, which more than likely is negative. The (expected) payoff will be the probability that the fake bid wins the game, multiplied by the amount he pays.

Putting (1) and (2) together, the total expected payoff is the probability the fake bidder can withdraw her bid multiplied by the amount from (1) plus the probability she cannot withdraw her bid, multiplied by the amount from (2). Therefore, we can write the probability of winning the auction as:

$$Prob(max_{j \neq i} b_j^c(\theta_j) <= b_i^c(\theta_i)) = F^*(b_i(\theta_i)) \quad for \quad i \in 1, ..., n, \quad i \neq j$$

where $F^*(x)$ is as defined above.

To sum up, we have $n$ players in the auction; each has their private valuation $\theta_i$ drawn from interval $[0, \infty)$ with the distribution function F(x), their bidding function is $b_i(\theta_i)$, and their cheating bid function for their hired fake bidders is

$b_i^c(\theta_i)$.

Withdrawing the cheating bid would be possible with the probability $p^w$. So, to formulate the payoff function of a player who would cheat in an English auction, we have:

$$E_i = p^w(F^*(b_i^c(\theta_i))(\theta_i - b_i(\theta_i))) + (1 - p^w)(F^*(b_i^c(\theta_i))(\theta_i - b_i^c(\theta_i)) \qquad (3.1)$$

Let us assume that:

1) All other players are of the same type as player $i$. Hence, they all have the option to cheat.

2) It is not rational for a cheating player to submit a fake bid that is less than or equal to his actual (honest) bid. So, we can assume for all $i$, $b_i^c(\theta_i) \geq b_i(\theta_i)$.

3) If the hired fake bidder wins the game and can withdraw her bid, the associated employer has no incentive to submit an honest bid more than the seller's reservation price. This is because when he is the only remaining player in the game, whatever he bids, he will win, and he has to pay it. So, as the seller's reservation price is the minimum amount he can pay, the optimal bid will be $b(\theta) = a$.

Considering the valuation of each player is distributed over $[0, \infty)$, we can write the payoff function as below:

$$E_i = p^w(F^*(b_i^c(\theta_i))(\theta_i - a)) + (1 - p^w)(F^*(b_i^c(\theta_i))(\theta_i - b_i^c(\theta_i)) \qquad (3.2)$$

To simplify the calculation of the model and ease of analysis, we assume $\theta_i$, for $i = 1, 2, ..., n$ are independent and identically distributed uniform random variables, drawn from $[0, 1]$. We are using the uniform distribution because we know that players' private valuations are drawn from $[0, 1]$, but we do not know exactly how they are distributed. So, we will give each value in this interval the same probability as others, so the result is not biased. Therefore, it would be reasonable to assume they are all distributed on $[0, 1]$ with equally likely probability. Thus we need a distribution that does distribute values with this probability for us. Therefore, we assume $\theta_i$ for $i = 1, 2, ..., n$ distributed uniformly on $[0, 1]$

By "identical" assumption, we suggest players are all the same, meaning they think the same way and play the same strategy. Therefore, we can search for symmetric NEs; in particular, we will search for symmetric linear NEs. This means NEs in which all players $i$ use a bidding strategy $b_i^c(\theta_i)$ which is linear in $\theta_i$.

Finally, by "independence," we mean each player gains no extra information about other players' private values given knowledge of his own private value.

## 3.2 The Analysis

Let us assume we are considering the case that: There are $n$ players in the game and all players are playing:

$$b^c(\theta) = \begin{cases} a + \alpha(\theta - a) & \text{if } a \leq \theta \leq 1 \\ \\ 0 & \text{if } \theta < a \end{cases} \tag{3.3}$$

The question is, for what value of $\alpha$ will $(b_i(\theta), b_{-i}(\theta))$ be a NE?

Consider player one, his expected payoff function, as we discussed, is a function of his private value $\theta_1$, $b_1^c(\theta_1)$, and the probability of withdrawing his bid $p^w$. For ease, first let's consider the seller's reservation price in the game is zero, so we set $a = 0$ and the strategy for player one will change to:

$$b_1^c(\theta_1) = \alpha\theta_1 \qquad \text{for } 0 \leq \theta_1 < 1 \tag{3.4}$$

### 3.2.1 $|N| = 2$

Let us consider the case where there are two players. We have:

$$\begin{aligned} F^*(x) &= P(b_2^c \leq x) = P(\alpha\theta_2 \leq x) \\ &= P(\theta_2 \leq \frac{x}{\alpha}) = \frac{x}{\alpha} \end{aligned} \tag{3.5}$$

Hence, substituting into (3.2) we have:

$$E_1 = p^w(\frac{(b_1^c(\theta_1))}{\alpha})(\theta_1 - 0) + (1 - p^w)(\frac{(b_1^c(\theta_1))}{\alpha})(\theta_1 - b_1^c(\theta_1)) \tag{3.6}$$

Player one tries to maximize $E_1$ over $b_1^c \in [\theta_1, 1]$. To maximize this function, we would take its derivative with respect to $b_1^c(\theta_1)$ and set it equal to zero. In order to have a symmetric linear Nash equilibrium, the maximizer $b_1^{c*}(\theta_1)$ should be equal to $\alpha * \theta_1$ for some $\alpha$.

The derivative of the function would be equal to:

$$\frac{\partial E_1}{\partial b_1^c} = \frac{(1-p^w).(\theta_1 - 2b_1^c(\theta_1))}{\alpha} + \frac{p^w.\theta_1}{\alpha} = \frac{\theta_1 + 2p^w b_1^c(\theta_1) - 2b_1^c(\theta_1)}{\alpha} \tag{3.7}$$

Set it equal to zero to find:

$$b_1^{c*}(\theta_1) = \frac{\theta_1}{2(1-p^w)} \tag{3.8}$$

Now, to find the value of $\alpha$ that makes this strategy an NE, we set $b_i^{c*}(\theta_1) = \alpha * \theta_1$ and we solve it for $\alpha$

$$b_1^c(\theta_1) = \alpha * \theta_1 \Rightarrow \frac{\theta_1}{2(1-p^w)} = \alpha\theta_1 \Rightarrow \alpha = \frac{1}{2(1-p^w)} \tag{3.9}$$

Now that we have the optimal value for $\alpha$, we can have the NE strategy for player one as follows:

$$b_1^c(\theta_1) = \frac{\theta_1}{2(1-p^w)} \qquad \text{for} \quad 0 \leq \theta_1 \leq 1 \tag{3.10}$$

Now we want to check if this strategy is a Nash equilibrium. We must make sure player two will choose this strategy, given that player one plays the same strategy. Assuming both players are following strategy (3.10), the expected payoff function for player two will be:

$$E_2 = p^w(\frac{(b_2^c(\theta_2))}{2(1-p^w)})(\theta_2 - 0) + (1-p^w)(\frac{(b_2^c(\theta_2))}{2(1-p^w)})(\theta_2 - b_2^c(\theta_2)) \tag{3.11}$$

Finding the optimal bid for player two, we first will have the derivative of the expected payoff function with respect to $b_2^c(\theta_2)$:

$$\begin{aligned} \frac{\partial E}{\partial b_2^c(\theta_2)} &= \frac{(1-p^w).(\theta_2 - 2b_2^c(\theta_2))}{2(1-p^w)} + \frac{p^w.\theta_2}{2(1-p^w)} \\ &= \frac{\theta_2 + 2p^w b_2^c(\theta_2) - 2b_2^c(\theta_2)}{2(1-p^w)} \end{aligned} \tag{3.12}$$

Set it equal to zero and solve it for $b_i^c(\theta)$ we will have:

$$\begin{aligned} \frac{\theta_2 + 2p^w b_2^c(\theta_2) - 2b_2^c(\theta_2)}{2(1-p^w)} = 0 &\Rightarrow \theta_2 + 2p^w b_2^c(\theta_2) - 2b_2^c(\theta_2) = 0 \\ &\Rightarrow \theta_2 = 2b_2^c(\theta_2) - 2p^w b_2^c(\theta_2) = 2b_2^c(\theta_2)(1-p^w) \\ &\Rightarrow b_2^{c*}(\theta_2) = \frac{\theta_2}{2(1-p^w)} \end{aligned} \tag{3.13}$$

As we can see, the optimal strategy for player two is the same as what was given as player one optimal strategy; therefore, it is Nash equilibrium.

Thus, we can write our result as the following proposition.

**Proposition 3.2.1** In a two players English auction, where there is a probability $p^w$ that players can withdraw their bid, it is a Nash equilibrium for the cheating players to bid according to:

$$b_i^c(\theta_i) = \frac{\theta_i}{2(1-p^w)} \tag{3.14}$$

It is also notable that as a player's value of the item increases, i.e., $\theta_i$ goes up, $b_i^c(\theta_i)$ also increases. Thus, we can conclude that $b_i^c(\theta_i)$ is also an increasing function. Furthermore, as $p^w$ goes up, the denominator gets smaller, so the fraction gets larger, $b_i^c(\theta_i)$ goes up. Our intuition can confirm that since it is more likely that a player can withdraw his bid, it is safer to cheat. They can therefore take a higher risk and bid higher.

### 3.2.2 $|N| = 3$

Now let us see what would happen if we have three players in the game. First of all, we consider all of the players all playing the same strategy:

$$b_i^c(\theta_i) = \alpha\theta_i \qquad \text{for} \quad 0 \le \theta_i \le 1 \tag{3.15}$$

for some value of $\alpha$, where $i = 1, 2, 3$. Therefore, the expected payoff function for player $i$, $i = 1, 2, 3$ will be:

$$E_i = p^w(\frac{b_i^c(\theta_i)}{\alpha})^2(\theta_i - 0) + (1-p^w)(\frac{b_i^c(\theta_i)}{\alpha})^2(\theta_i - b_i^c(\theta_i)) \tag{3.16}$$

Here again, we have two terms; one is when the fake bidder can withdraw her bid with probability $p^w$, and her submitted bid is higher than the other two bids with probability $(\frac{b_i^c(\theta_i)}{\alpha})(\frac{b_i^c(\theta_i)}{\alpha})$. Thus the player will win and has to pay his actual bid, which is equal to the reservation price minus zero. The second term is when the fake bidder has to stay in the game with a probability of $(1 - p^w)$, and is the winner of the game with probability $(\frac{b_i^c(\theta_i)}{\alpha})(\frac{b_i^c(\theta_i)}{\alpha})$

Using the same way as we did for the case $|N| = 2$ to find optimal $b_i^c(\theta)$, we will have:

$$\begin{aligned} \frac{\partial E}{\partial b_i^c(\theta_i)} &= \frac{(1-p^w)(2\theta_i b_i^c(\theta_i) - 3b_i^c(\theta_i)^2)}{\alpha^2} + \frac{2p^w\theta_i b_i^c(\theta_i)}{\alpha^2} \\ &= \frac{b_i^c(\theta_i)(2\theta_i - 3b_i^c(\theta_i) + 3p^w b_i^c(\theta_i))}{\alpha^2} \end{aligned} \tag{3.17}$$

Set $\frac{\partial E}{\partial b_i^c(\theta_i)} = 0$

$$\frac{b_i^c(\theta_i)(2\theta_i - 3b_i^c(\theta_i) + 3p^w b_i^c(\theta_i))}{\alpha^2} = 0 \tag{3.18}$$

This gives two critical values for $b_i^c(\theta_i)$; $b_i^c(\theta_i) = \frac{2\theta_i}{3(1-p^w)}$ and $b_i^c(\theta_i) = 0$. However, using "first derivative test" we can see $b_i^c(\theta_i) = 0$ is not an optimal answer here, as shown in Figure 3.1.

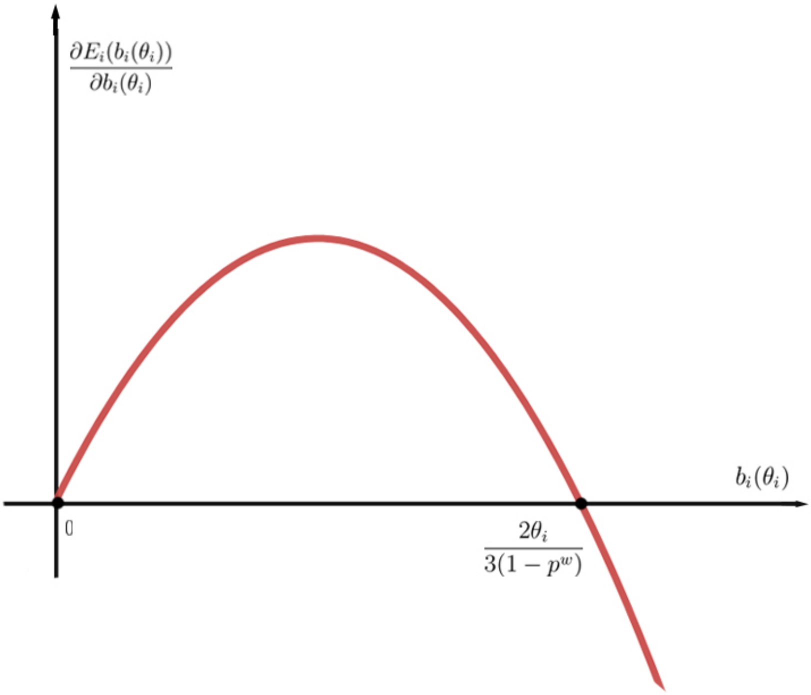


Figure 3.1: Derivative of the Expected Payoff Function $E_i$

According to the plot, at $b_i^c(\theta_i) = 0$, the function $E_i$ changing from decreasing to increasing, therefore, $b_i^c(\theta_i) = 0$ is a local minimum, not a local maximum. So, it cannot be an optimal value for the expected payoff function. Thus, the only optimal value for $b_i^c(\theta_i)$ will be:

$$b_i^c(\theta_i) = \frac{2\theta_i}{3(1-p^w)} \tag{3.19}$$

Having the value of $b_i^c(\theta_i)$, we can find the optimal value for $\alpha$ to write the optimal strategy for each player in the game.

$$\begin{aligned} b_i^c(\theta_i) = \alpha * \theta_i \Rightarrow \frac{2\theta_i}{3(1-p^w)} &= \alpha\theta_i \\ \Rightarrow \alpha &= \frac{2}{3(1-p^w)} \end{aligned} \tag{3.20}$$

Therefore,

$$b_i^c(\theta_i) = \frac{2\theta_i}{3(1-p^w)} \quad \text{if} \quad 0 \le \theta_i \le 1 \tag{3.21}$$

Now, let us see if this is a Nash equilibrium here too. Let us suppose player one assume that all other players are following this strategy, so his optimal strategy can be calculated as follows:

$$E_1 = p^w(\frac{(b_1^c(\theta_1))}{\frac{2}{3(1-p^w)}})^2(\theta_1 - 0) + (1-p^w)((\frac{(b_1^c(\theta_1))}{\frac{2}{3(1-p^w)}})^2)(\theta_1 - b_1^c(\theta_1)) \tag{3.22}$$

$$\begin{aligned} \frac{\partial E}{\partial b_1^c(\theta_1)} &= \frac{9p^w\theta(1-P_w)^2(2b_1^c(\theta_1))}{4} + \frac{9(1-p^w)^3(2\theta b_1^c(\theta_1) - 3b_1^c(\theta_1)^2)}{4} \\ &= \frac{9b_1^c(\theta_1)(1-p^w)^2(2\theta - 3b_1^c(\theta_1)(1-p^w))}{4} \end{aligned} \tag{3.23}$$

Set $\frac{\partial E}{\partial b_1^c(\theta_1)} = 0$

$$\begin{cases} \frac{9b_1^c(\theta_1)(1-p^w)^2}{4} = 0 & b_i^c(\theta_i) = 0 \\ 2\theta - 3b_1^c(\theta_1)(1-p^w)) = 0 & b_i^c(\theta_i) = \frac{2\theta_i}{3(1-p^w)} \end{cases} \tag{3.24}$$

With the same logic as before, we know that $b_i^c(\theta_i) = 0$ is not an optimal answer for $b_i^c(\theta_i)$, therefore, $b_i^c(\theta_i) = \frac{2\theta_i}{3(1-p^w)}$ is the only optimal answer, which makes the suggested strategy a Nash equilibrium.

### 3.2.3 $N$ general

Calculating the Nash equilibrium for games with $N = 2, 3, 4, ...$ players gives us the idea that we can generalize the optimal strategy for this game with $n$ players.

**Theorem:** Suppose we have $n$ bidders in an English Auction and their private values of the item being sold $\theta_i$, for $i = 1, 2, .., n$, are each uniformly distributed over [0,1]. Suppose each player hires one fake bidder in the game. Then, it is a Nash equilibrium for each player $i$ to follow the strategy below.

$$b_i^c(\theta) = \frac{(n-1)\theta_i}{n(1-p^w)} \quad \text{if} \quad 0 \le \theta_i \le 1 \tag{3.25}$$

**Proof:** At each point, the game-winner will be the player whose bid is higher than others. Let us assume we have $n$ players in the game. Suppose each player brings one fake bidder to the game. If a player decides not to cheat, his associated fake bidder will submit a bid equal to his bid. Therefore, we have $n$ actual players in the game and $n$ fake bidders, which adds up to $2n$ players. However, there are only $n$ decision-makers in the game, each responsible for two bids, one their own and one their associated fake bids. Thus, as the fake bidders' submitted bids will always be greater than or equal to the associated player's actual bid, the competition for the highest bid will be between $n$ bids. As a result, for a fake player to be the winner of the game, her bid must be higher than $(n-1)$ other bids, so the expected profit function can be written as follows:

$$E_i = p^w(\frac{b_i^c(\theta_i)}{\alpha})^{(n-1)}(\theta_i - 0) + (1-p^w)(\frac{b_i^c(\theta_i)}{\alpha})^{(n-1)}(\theta_i - b_i^c(\theta_i)) \quad (3.26)$$

Using the same logic as before, we can calculate the optimal $b_i^c(\theta_i)$:

$$\begin{aligned}\frac{\partial E}{\partial b_i^c(\theta_i)} &= \frac{(1-p^w).((n-1)\theta_i b_i^c(\theta_i)^{n-2} - nb_i^c(\theta_i)^{n-1})}{\alpha^{n-1}} + \frac{(n-1)p^w.\theta_i b_i^c(\theta_i)^{n-2}}{\alpha^{n-1}} \\ &= \frac{b_i^c(\theta_i)^{n-2}((n-1)\theta_i - nb_i^c(\theta_i) + np^w b_i^c(\theta_i))}{\alpha^{n-1}}\end{aligned} \quad (3.27)$$

Set $\frac{\partial E}{\partial b_i^c(\theta_i)} = 0$

$$\frac{b_i^c(\theta_i)^{n-2}((n-1)\theta_i - nb_i^c(\theta_i) + np^w b_i^c(\theta_i))}{\alpha^{n-1}} = 0 \quad (3.28)$$

This gives us two critical values for $b_i^c(\theta_i)$; $b_i^c(\theta_i) = \frac{(n-1)\theta_i}{n(1-p^w)}$ and $b_i^c(\theta_i) = 0$.

Therefore, following the same logic as in section (3.2.2), the optimal value for $b_i^c(\theta_i)$ will be:

$$b_i^c(\theta_i) = \frac{(n-1)\theta_i}{n(1-p^w)} \quad (3.29)$$

Using equation (3.28), the $\alpha$ that makes the strategy optimal will be:

$$\alpha = \frac{(n-1)}{n(1-p^w)} \quad (3.30)$$

So, we can write the final strategy as:

$$b_i^c(\theta_i) = (\frac{n-1}{n(1-p^w)})\theta_i \quad \text{if} \quad 0 \leq \theta_i \leq 1 \tag{3.31}$$

This is a Nash equilibrium because if player $i, i = 1, 2, ..., n$ is given that all other $n-1$ players in the game are using this strategy, his optimal strategy will also be the same. This can be concluded after the following calculation to find the optimal strategy for player $i$.

$$E_i = p^w(\frac{b_i^c(\theta_i)}{\frac{n-1}{n(1-p^w)}})^{n-1}(\theta_i - 0) + (1-p^w)(\frac{b_i^c(\theta_i)}{\frac{n-1}{n(1-p^w)}})^{n-1}(\theta_i - b_i^c(\theta_i)) \tag{3.32}$$

$$\begin{aligned}\frac{\partial E}{\partial b_i^c(\theta_i)} &= \frac{(n(1-p^w))^{n-1}p^w\theta((n-1))}{(n-1)^{(n-1)}} \\ &+ \frac{(n(1-p^w)(^{n-1}(1-p^w)((n-1)\theta b_i^c(\theta_i)^{n-2} - b_i^c(\theta_i)^{n-1})}{(n-1)^{(n-1)}} \\ &= \frac{n^{n-1}(1-p^w)^{n-1}b_i^c(\theta_i)^{n-1}((n-1)\theta - n(1-p^w)b_i^c(\theta_i))}{(n-1)^{(n-1)}}\end{aligned} \tag{3.33}$$

Set $\frac{\partial E}{\partial b_i^c(\theta_i)} = 0$

$$\frac{n^{n-1}(1-p^w)^{n-1}b_i^c(\theta_i)^{n-1}((n-1)\theta - n(1-p^w)b_i^c(\theta_i))}{(n-1)^{(n-1)}} = 0 \tag{3.34}$$

$$\begin{cases} \frac{n^{n-1}(1-p^w)^{n-1}b_i^c(\theta_i)^{n-1}}{(n-1)^{(n-1)}} = 0 & b_i^c(\theta_i) = 0 \\ (n-1)\theta - n(1-p^w)b_i^c(\theta_i) = 0 & b_i^c(\theta_i) = \frac{(n-1)\theta_i}{n(1-p^w)} \end{cases} \tag{3.35}$$

As discussed in the previous section, $b_i^c(\theta_i) = 0$ cannot be an optimal value here because it is not a local maximum. Therefore, the only optimal strategy here is $b_i^c(\theta_i) = \frac{(n-1)\theta_i}{n(1-p^w)}$, which is equal to the other players' best strategies, thus this is a Nash equilibrium. **QED**

As $p^w$ approaches zero, meaning bidders cannot withdraw the fake bid, it is rational to bid an amount to make a nonnegative profit if they win the fair game. In an English auction, if the possibility of cheating is almost zero, the setting is similar to the first-price auction in which the one with the highest bid will win and has to pay an amount he has submitted. Therefore, the above equation should give a strategy equal to the optimal strategy in the first-price auction.

We can verify it by setting $p^w = 0$:

$$\lim_{p^w \to 0} b_i^c(\theta_i) = (\frac{n-1}{n})\theta_i \quad \text{if} \quad 0 \leq \theta_i \leq 1 \tag{3.36}$$

which is equal to the strategy explained in Proposition (1.4.1).

Now consider the case where $p^w = 0$ and $n \to \infty$. Applying (3.31) gives $b_i^c(\theta_i) = \theta_i$, i.e. the optimal strategy for each player would be to bid an amount equal to his private valuation.

This suggests that cheating does not benefit anyone when there are infinite players in the game. This is true because if player $i$, with private value $\theta_i$, submits a bid less than $\theta_i$, there is always another player, with approximately the same valuation of the good, who can outbid player $i$ by an infinitesimal $\epsilon$. As a result, player $i$ will not win the game, and the payoff will be zero. On the other hand, if he submits a bid greater than $\theta_i$, he will receive a negative payoff in case of winning - which is not optimal. Therefore, there is no incentive for a player to bid more than his private value in this setting. Thus, it is a dominant strategy to bid $\theta_i$ for player $i$, $i \in 1, 2, \ldots, n$.

Now suppose $p^w$ is approaching one, meaning the fake bidders can safely cheat in the game and withdraw their bids. Let us again assume that $n$ can be any value. In this case, as there is no risk that the actual player will be forced to pay the amount his associated fake bidder has submitted in case of winning, the fake bidder can bid as high as she wants to increase the chance of winning. We can verify this by letting $p^w$ approach one in (3.31), obtaining:

$$\lim_{p^w \to 1} \frac{(n-1)\theta_i}{n(1-p^w)} = +\infty \tag{3.37}$$

### 3.2.4 Non-zero Reservation Price

Let us study the optimal strategy for this scenario with "$a$" as the game's seller's reservation price (Above, we considered the case where a=0). Again, we assume we have $|N| = 2$ players, and both players have $\theta_i, i \in \{1, 2\}$ uniformly distributed on $[0, 1]$. If a player's private value is less than or equal to "$a$," he bids 0 because bidding a positive amount cannot give him profit.

Thus, we limit our strategies (for both players) to those of the type above. as stated in (3.3):

$$b^c(\theta) = \begin{cases} a + \alpha(\theta - a) & \text{if } a \leq \theta \leq 1 \\ \\ 0 & \text{if } \theta < a \end{cases} \tag{3.38}$$

Therefore, the bidding function is a linear (affine) function of $\theta$ which is equal to "$a$" if $\theta = a$, and is equal to $a + \alpha(1-a)$ when $\theta = 1$. Accordingly, submitted bids are distributed on $[a, a + \alpha(1-a)]$

Then:

$$\begin{aligned} F^*(x) &= p(b_2^c(\theta_2) \leq x) = p(a + \alpha(\theta_2 - a) \leq x) \\ &= p(\alpha(\theta_2 - a) \leq x - a) = p(\theta_2 - a \leq \frac{x-a}{\alpha}) \\ &= \frac{x-a}{\alpha(1-a)} \end{aligned} \tag{3.39}$$

Thus, the profit function for player one is:

$$\begin{aligned} E_1 &= a * a * 0 \\ &+ (1-a) * a * [p^w(\theta_1 - a) + (1-p^w)(\theta_1 - b_1^c(\theta_1))] + a * (1-a) * 0 \\ &+ (1-a) * (1-a) * [p^w F^*(b_1^c(\theta_1))(\theta_1 - a) + (1-p^w)F^*(b_1^c(\theta_1)(\theta_1 - b_1^c(\theta_1))] \\ &= (1-a) * a[p^w(\theta_1 - a) + (1-p^w)(\theta_1 - b_1^c(\theta_1))] \\ &+ (1-a)^2[p^w(\frac{(b_1^c(\theta_1) - a)}{\alpha(1-a)})(\theta_1 - a) + (1-p^w)(\frac{(b_1^c(\theta_1) - a)}{\alpha(1-a)})(\theta_1 - b_1^c(\theta_1))] \end{aligned} \tag{3.40}$$

As stated above, players' private values are distributed on $[0, 1]$. Therefore, $\theta_i$ for $i \in 1, 2, ..., n$ is less than "$a$" with the probability $\frac{a-0}{1-0} = a$. Similarly, $\theta_i$ will be more than "$a$" with the probability $1 - a$. Thus, with probability $a^2$ both players' private values are less than or equal to "$a$", i.e., $\theta_1 \leq a$ and $\theta_2 \leq a$. With probability $a(1-a)$, $\theta_1 \leq a$ and $\theta_2 \geq a$; similarly, with probability $(1-a)a$, $\theta_1 \geq a$ and $\theta_2 \leq a$. Finally, with probability $(1-a)^2$, both $\theta_1$ and $\theta_2$ are greater than $a$, i.e., $\theta_1 > a$ and $\theta_2 > a$.

Let us look at the profit function from the perspective of player one. With probability $a^2$, both players have private valuations less than $a$, so they both submit zero; therefore, player 1's profit function will be 0. With probability $a(1-a)$, player one has a value more than $a$, and player two has a value less than $a$. Thus, it is inevitable that player two will bid 0 and player one will win, though we are unsure how much they will win. Similarly, with probability $(1-a)a$, player one has a value less than $a$, and player two has a value more than $a$.In this third case, player one will win zero, and finally, with probability $(1-a)(1-a)$, both players have values more than $a$, so player one's win depends on whether $\theta_1 > \theta_2 \geq a$ or $\theta_2 > \theta_1 \geq a$. The probability of this event has been discussed in section (3.2.1).

If he can cheat successfully, i.e., withdraw the fake bid, player one can obtain a profit equal to $(\theta_1 - a)$. However, if his associated fake bidder cannot withdraw her bid, then he has to pay $b_1^c(\theta_1)$, so the profit will be $(\theta_1 - b_1^c(\theta_1))$. Thus, although player one would win the game if he does not cheat, he would still benefit $(\theta_1 - a)$ by cheating if he could withdraw the fake bid.

Lastly, when both player's private values are greater than "$a$" with the probability $(1-a)$, there is a probability $\frac{b-a}{\alpha}$ that player one will beat player two. Therefore, he has to pay $(\theta_1 - a)$ if his fake bidder can withdraw her bid, and $(\theta_1 - a)$ otherwise.

To find the optimal $b^c$, following the procedure in the last section, we take the partial derivative of $E$ with respect to b and set it equal to zero. The result would be:

$$\frac{\partial E_1}{\partial b_1^c(\theta_1)} = 0 - (1-a)a(1-p^w) + (1-a)(\frac{p^w(\theta_1 - a) + (1-p^w)(\theta_1 + a - 2b_1^c(\theta_1)}{\alpha}) \tag{3.41}$$

Set it equal to zero and solve it for $b_1^c(\theta_1)$ we will have:

$$\frac{\partial E_1}{\partial b_1^c(\theta_1)} = 0 - (1-a)a(1-p^w) + (1-a)(\frac{p^w(\theta_1 - a) + (1-p^w)(\theta_1 + a - 2b_1^c(\theta_1)}{\alpha}) = 0 \tag{3.42}$$

$$\begin{aligned} \Rightarrow b_1^c(\theta_1) &= \frac{a + \theta_1 - a\alpha}{2} + \frac{p^w(\theta_1 - a)}{2(1-p^w)} \\ &= \frac{a + \theta_1 - 2p^w a - a\alpha(1-p^w)}{2(1-p^w)} \end{aligned} \tag{3.43}$$

Having optimal $b_1^c(\theta_1)$, we can solve the $b_i^c(\theta_i) = a + \alpha * \theta_i$ equation to find the value of $\alpha$ and it can tell each player bidding what portion of his value makes $b(\theta)$ a Nash equilibrium for each player.

Set $b_i^c(\theta_i) = a + \alpha * (\theta_i - a)$

$$\begin{aligned} \frac{a + \theta_1 - 2p^w a - a\alpha(1-p^w)}{2(1-p^w)} &= a + \alpha * (\theta_i - a) \\ \Rightarrow \alpha &= \frac{a - \theta_i}{(2\theta_1 - a)(1-p^w)} \end{aligned} \tag{3.44}$$

It is aligned with our intuition since if we set $a = 0$, we will get (3.9) in section (3.2.1) where we had two players in the game, and the reservation price was zero. Therefore, we can have:

$$b_1^c(\theta) = \begin{cases} a + \frac{(\theta_1 - a)^2}{(2\theta_1 - a)(1 - p^w)} & \text{if } a \leq \theta_1 \leq 1 \\ 0 & \text{if } \theta_1 < a \end{cases} \tag{3.45}$$

However, this is problematic since the optimal $b_i^c(\theta)$ in equation (3.45) is not of the linear form in equation (3.38). Hence, there are no symmetric linear Nash equilibrium unless $a = 0$

## 3.3 Other Distributions

In addition to the uniform distribution, players' private value can be distributed on their interval by different distribution functions. The various distributions indicate players' different opinions about the item's value they are bidding on. For instance, if $\theta_i$, for $i = 1, 2, ..., n.$, are distributed "exponentially" on $[0, \infty)$, it can be interpreted that players in the game mostly believe the item being sold is worth about zero dollars. The graph below (Figure 3.2) shows that private values are more concentrated around zero.

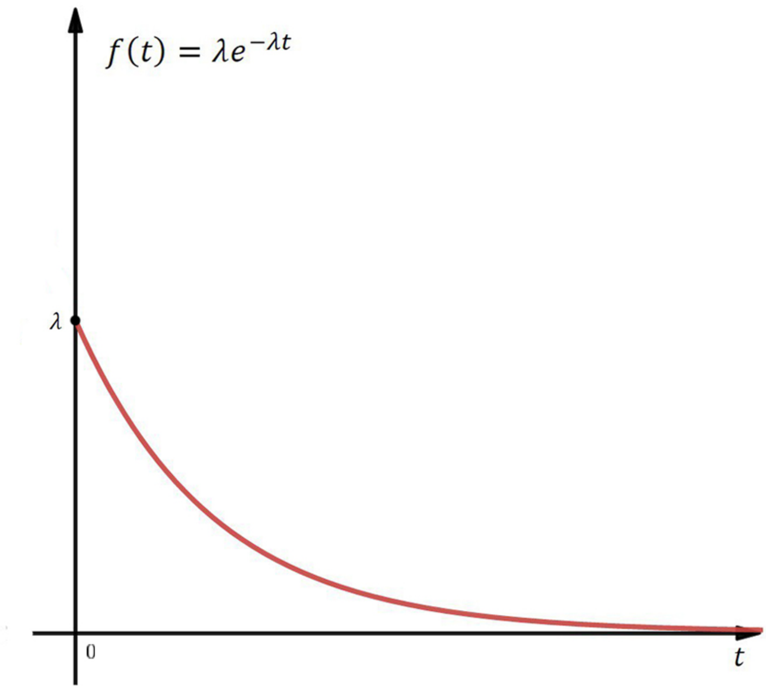


Figure 3.2: Exponential Density Function

In order to determine whether the Nash equilibrium is only dependent on uniform distribution or not, we followed the same process for three different distributions: exponential, geometrical, and an arbitrary distribution function. The goal is to see if there is any Nash equilibrium for players whose private values are distributed by one of these distribution functions on the same interval. In addition, we are trying to see if it is profitable for player $i$ to follow the strategy he believes all other players follow.

Below is the study on model behavior on the exponential distribution. Here as before in section (3.1), we believe players will bid according to this function. Their bid is a function of their private values, and we assumed that $a = 0$ for ease.

$$b(\theta) = \begin{cases} \alpha\theta & \text{if } 0 < \theta < 1 \\ \\ 0 & \text{if } \theta \leq 0 \end{cases} \tag{3.46}$$

Therefore, the profit function will be again:

$$E = p^w Prob(b^c(\theta_i) > b^c(\theta_{-i}))(\theta_i - a) + (1 - p^w) Prob(b^c(\theta_i) > b^c(\theta_{-i}))(\theta_i - b^c(\theta_i)) \tag{3.47}$$

To find the probability of being the highest submitted bid for each bid, we have to examine $b(\theta)$. If the player $i$ values the item positively, $\theta_i > 0$, he would bid $b(\theta_i) = \alpha\theta_i$. If player $i$ bids $b(\theta_i)$, the probability that he wins is the probability that $b(\theta_i) > b(\theta_{-i})$. For a two-player game, this probability for player one will be equal to the probability that $b(\theta_1) \geq \alpha\theta_2$. Therefore, $\theta_2 \leq \frac{b(\theta_1)}{\alpha}$.

Thus, when private values are distributed exponentially, given the density function is $f(t) = \lambda e^{-\lambda t}$, the probability that player one has the highest bid can be calculated by:

$$Prob(b(\theta_1) > b(\theta_2)) = \int_0^{\frac{b(\theta_1)}{\alpha}} \lambda e^{-\lambda t} dt = [1 - e^{-\lambda t}]_0^{\frac{b}{\alpha}} = 1 - e^{-\lambda \frac{b(\theta_1)}{\alpha}} \tag{3.48}$$

Even though the probability for a uniform distribution from previous section was apparent (equation (3.5)), it can also be calculated using the above method.

$$Prob(b(\theta_1) > b(\theta_2)) = \int_0^{\frac{b(\theta_1)}{\alpha}} 1 dt = \frac{b(\theta_1)}{\alpha} \tag{3.49}$$

Back to exponential distribution, using the result from equation (3.48), we can write the profit function for player one if he intends to commit cheating.

$$E_1 = p^w(1 - e^{-\lambda \frac{b(\theta_1)}{\alpha}})(\theta_1 - 0) + (1 - p^w)(1 - e^{-\lambda \frac{b(\theta_1)}{\alpha}})(\theta_1 - b_1^c(\theta_1)) \tag{3.50}$$

Following the same procedure as in section (3.1), after solving for $\alpha$, we will end up with a nonlinear strategy for player one while $b(\theta_i) = \alpha\theta_i$ is a linear function. This means that players cannot achieve their best profit by following the same strategy; therefore, there is no Nash equilibrium in this case. The results are the same for the geometric distribution with density function $f(t) = q(1-q)t-1$ and the arbitrary distribution with density function $f(t) = 2(1-t)$.

# Chapter 4

# Conclusion

## 4.1 Results

In this thesis, after introducing auction environments, auction types, private valuation, and bidding strategies, we have introduced some tools that game theory has provided to help model scenarios in real-life auction transactions as a mathematical problem and find a solution to it. After that, we have proposed a scenario in English auction with certain rules, and based on non-cooperative game concepts, constructed a model based on related variables. These variables consisted of players' private values of the item, the probability of being able to withdraw their submitted bids, and the reservation price set by the seller at the start of the game. The next step was to modify the scenario and check if the model works in each of them. In doing so, we realized that the model follows the well-known results in auction theory.

Our suggested model has been verified through working in main settings in well-known auctions. When the probability of withdrawing bids is approaching zero, the game setting reverts to a first-price auction. Therefore, by setting $p^w = 0$ in the model, we have achieved the optimal strategy of the first-price auction. This also suggests that cheating does not benefit anyone when there are infinite players in the game. Moreover, the model has verified the case that when there is no risk for players to cheat, they find it optimal to bid as high as they want.

## 4.2 Future Work

As we examined how the model did not result with the expected conclusion in geometric and exponential distributions in section (3.3), we find it important to find a pattern between different distributions to see if the model draws out expected results in them. It is also essential to continue modifying the model in a way that works on every general distribution if possible.

As discussed in section (3.1), we have assumed that all players are of the same kind, and they are all willing to cheat in the game. However, there is an interesting scenario in which a subset of players is honest. This means that not all players will hire a fake bidder; therefore, modeling this situation will be more complicated.

Another scenario that we find interesting to work on in the future is to have a penalty (fixed, a percentage of their submitted bid, etc.) for cheating if they get caught and a probability for getting caught during the game. Therefore, players have to choose between cheating and having the item at a lower price, getting caught while cheating and being forced to pay the penalty, or playing honestly!

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